

Fractional Discrete Systems with Sequential h -differences

Ewa Girejko, Dorota Mozyrska, Małgorzata Wyrwas

Faculty of Computer Science

Department of Mathematics

Białystok University of Technology

Wiejska 45A, 15-351 Białystok, Poland

{e.girejko, d.mozyrska, m.wyrwas}@pb.edu.pl

Abstract

In the paper we study the subject of positivity of systems with sequential fractional difference. We give formulas for the unique solutions to systems in linear and semi-linear cases. The positivity of systems is considered.

1 Introduction

The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19-th century. Later on, in the late 1960s, this idea was used by engineers for modeling various processes. Thus the fractional calculus started to be exploited since that time. This calculus is a field of mathematics that grows out of the traditional definitions of calculus integral and derivative operators and deals with fractional derivatives and integrals of any order. Fractional difference calculus have been investigated by many authors, for example, [1, 3, 4, 6, 8, 10, 12, 14–17] and others. The subject of positivity is well developed for fractional linear systems with continuous time, see [11–13]. However, positivity of fractional discrete systems with sequential h -differences is still a field to be examined. In the present paper we open our studies in this field. We give formulas for the unique solutions to systems in linear and semi-linear cases. Moreover, the positivity of systems is considered.

The paper is organized as follows. In Section 2 all preliminaries definitions, facts and notations are gathered. Section 3 presents systems with sequential fractional differences with results on uniqueness of solutions. Semilinear systems we included in Section 4. The last Section concerns positivity of considered systems.

2 Preliminaries

Let us denote by \mathcal{F}_D the set of real valued functions defined on D . Let $h > 0, \alpha > 0$ and put $(h\mathbb{N})_a := \{a, a+h, a+2h, \dots\}$ for $h > 0$ and $a \in \mathbb{R}$. Let

$$R_{\geq}^n = \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n\}.$$

Due to notations from the time scale theory the operator $\sigma : (h\mathbb{N})_a \rightarrow (h\mathbb{N})_a$ is defined by $\sigma(t) := t + h$. The next definitions of h -difference operator was originally given in [5], but here we propose simpler notation.

Definition 1. For a function $x \in \mathcal{F}_{(h\mathbb{N})_a}$ the *forward h -difference operator* is defined as

$$(\Delta_h x)(t) := \frac{x(\sigma(t)) - x(t)}{h}, \quad t = a + nh, n \in \mathbb{N}_0,$$

while the *h -difference sum* is given by

$$({}_a\Delta_h^{-1}x)(t) := h \sum_{k=0}^n x(a + kh),$$

where $t = a + (n+1)h, n \in \mathbb{N}_0$ and $({}_a\Delta_h^{-1}x)(a) = 0$.

Definition 2. [5] For arbitrary $t, \alpha \in \mathbb{R}$ the *h -factorial function* is defined by

$$t_h^{(\alpha)} := h^\alpha \frac{\Gamma(\frac{t}{h} + 1)}{\Gamma(\frac{t}{h} + 1 - \alpha)}, \quad (1)$$

where Γ is the Euler gamma function, $\frac{t}{h} \notin \mathbb{Z}_- := \{-1, -2, -3, \dots\}$, and we use the convention that division at a pole yields zero.

Notice that if we use the general binomial coefficient $\binom{a}{b} := \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}$, then (1) can be rewritten as

$$t_h^{(\alpha)} = h^\alpha \Gamma(\alpha + 1) \binom{\frac{t}{h}}{\alpha}.$$

The next definition with another notations was stated in [5]. Here we use more suitable summations.

Definition 3. For a function $x \in \mathcal{F}_{(h\mathbb{N})_a}$ the *fractional h -sum of order $\alpha > 0$* is given by

$$({}_a\Delta_h^{-\alpha}x)(t) := \frac{h}{\Gamma(\alpha)} \sum_{k=0}^n (t - \sigma(a + kh))_h^{(\alpha-1)} x(a + kh),$$

where $t = a + (\alpha + n)h, n \in \mathbb{N}_0$. Moreover we define $({}_a\Delta_h^0 x)(t) := x(t)$.

Remark 4. Note that ${}_a\Delta_h^{-\alpha} : \mathcal{F}_{(h\mathbb{N})_a} \rightarrow \mathcal{F}_{(h\mathbb{N})_{a+\alpha h}}$.

Accordingly to the definition of h -factorial function the formula given in Definition 3 can be rewritten as:

$$\begin{aligned}({}_a\Delta_h^{-\alpha}x)(t) &= h^\alpha \sum_{k=0}^n \frac{\Gamma(\alpha+n-k)}{\Gamma(\alpha)\Gamma(n-k+1)} x(a+kh) \\ &= h^\alpha \sum_{k=0}^n \binom{n-k+\alpha-1}{n-k} x(a+kh)\end{aligned}$$

for $t = a + (\alpha + n)h$, $n \in \mathbb{N}_0$.

Remark 5. In [10] one can find the following form of the fractional h -sum of order $\alpha > 0$:

$$({}_a\Delta_h^{-\alpha}x)(t) = \frac{h^\alpha}{\Gamma(\alpha)} \sum_{k=a}^{t-\alpha h} \left(\frac{t-\sigma(k)}{h} \right)^{(\alpha-1)} x(k)$$

that can be useful in implementation.

The following definition one can find in [2] for $h = 1$.

Definition 6. Let $\alpha \in (0, 1]$. The *Caputo h -difference operator* ${}_a\Delta_{h,*}^\alpha x$ of order α for a function $x \in \mathcal{F}_{(h\mathbb{N})_a}$ is defined by

$$({}_a\Delta_{h,*}^\alpha x)(t) := \left({}_a\Delta_h^{-(1-\alpha)} (\Delta_h x) \right)(t), \quad t \in (h\mathbb{N})_{a+(1-\alpha)h}.$$

Remark 7. Note that: ${}_a\Delta_{h,*}^\alpha : \mathcal{F}_{(h\mathbb{N})_a} \rightarrow \mathcal{F}_{(h\mathbb{N})_{a+(1-\alpha)h}}$, where $\alpha \in (0, 1]$.

We need the power rule formulas in the sequel. Firstly, we easily notice that for $p \neq 0$ the well defined h -factorial functions have the following property:

$$\Delta_h(t-a)_h^{(p)} = p(t-a)_h^{(p-1)}.$$

More properties of h -factorial functions can be found in [9]. In our consideration the crucial role plays the power rule formula presented in [7], i.e.

$$({}_a\Delta_h^{-\alpha}\psi)(t) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} (t-a+\mu h)_h^{(\mu+\alpha)}, \quad (2)$$

where $\psi(r) = (r-a+\mu h)_h^{(\mu)}$, $r \in (h\mathbb{N})_a$, $t \in (h\mathbb{N})_{a+\alpha h}$. Note that using the general binomial coefficient one can write (2) as

$$({}_a\Delta_h^{-\alpha}\psi)(t) = \Gamma(\mu+1) \binom{n+\alpha+\mu}{n} h^{\mu+\alpha}.$$

Then if $\psi \equiv 1$, then we have for $\mu = 0$, $a = (1-\alpha)h$ and $t = nh + a + \alpha h$

$$\begin{aligned}({}_a\Delta_h^{-\alpha}1)(t) &= \frac{1}{\Gamma(\alpha+1)} (t-a)_h^{(\alpha)} \\ &= \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)} h^\alpha = \binom{n+\alpha}{n} h^\alpha.\end{aligned}$$

Let us define special functions, that we use in the next section to write the formula for solutions.

Definition 8. For $\alpha, \beta > 0$ we define

$$\varphi_{k,s}(nh) := \begin{cases} \binom{n-k+k\alpha+s\beta}{n-k} h^{k\alpha+s\beta}, & \text{for } n \in \mathbb{N}_k \\ 0, & \text{for } n < k \end{cases} \quad (3)$$

and

$$\tilde{\varphi}_{k,s}(nh) := \begin{cases} \binom{n+\mu-1}{n} h^\mu = \frac{\Gamma(n+\mu)}{\Gamma(\mu)\Gamma(n+1)} h^\mu, & \text{for } n \in \mathbb{N}_0 \\ 0, & \text{for } n < 0 \end{cases}, \quad (4)$$

where $n, k, s \in \mathbb{N}_0$ and $\mu = k\alpha + s\beta$.

Remark 9. It is worthy to notice that:

- (a) $\varphi_{0,0}(nh) = 1$;
- (b) $\varphi_{1,0}(nh) = \binom{n+\alpha-1}{n-1} h^\alpha = ({}_0\Delta_h^{-\alpha} 1)((n-1)h + \alpha h)$;
- (c) $\varphi_{k,s}(nh) = \frac{\Gamma(n-k+1+k\alpha+s\beta)}{\Gamma(k\alpha+s\beta+1)\Gamma(n-k+1)}$ and as the division by pole gives zero, the formula works also for $n < k, n \in \mathbb{N}$;
- (d) $\varphi_{k,s}(nh) = \frac{1}{\Gamma(k\alpha+s\beta+1)} ((n-k)h + k\alpha h + s\beta h)_h^{(k\alpha+s\beta)}$.

We also need the property presented in the following proposition.

Proposition 10. Let $\alpha, \beta \in (0, 1], h > 0$ and $a = (\alpha - 1)h, b = (\beta - 1)h$. Then

$$({}_0\Delta_h^{-\alpha} \varphi_{k,s})(nh + a) = \varphi_{k+1,s}(nh) \quad (5)$$

and

$$({}_0\Delta_h^{-\beta} \varphi_{k,s})(nh + b) = \varphi_{k,s+1}(nh). \quad (6)$$

Proof. We show only equality (5), as (6) is a symmetric one.

Let $\mu := k\alpha + s\beta$. For $r \in (h\mathbb{N})_{kh}$ we define the following h -factorial function $\psi(r) := (r + \mu h)_h^{(\mu)}$. Since

$$\begin{aligned} \varphi_{k,s}(nh) &= \frac{1}{\Gamma(k\alpha + s\beta + 1)} ((n-k)h + k\alpha h + s\beta h)_h^{(k\alpha+s\beta)} \\ &= \frac{1}{\Gamma(\mu + 1)} \psi(nh - kh) \end{aligned}$$

for $n \geq k$ and $\varphi_{k,s}(mh) = 0$ for $m < k$, by (2) we get

$$\begin{aligned} ({}_0\Delta_h^{-\alpha} \varphi_{k,s})(t) &= ({}_{kh}\Delta_h^{-\alpha} \varphi_{k,s})(t) = \frac{1}{\Gamma(\mu + 1)} ({}_0\Delta_h^{-\alpha} \psi)(t) \\ &= \frac{1}{\Gamma(\mu + 1)} \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \alpha + 1)} (t + \mu h)_h^{(\mu+\alpha)} \\ &= \frac{1}{\Gamma(\mu + \alpha + 1)} (t + \mu h)_h^{(\mu+\alpha)}, \end{aligned}$$

where $t = a - kh + nh$. Hence

$$t + \mu h = nh - (k+1)h + (k+1)\alpha h + s\beta h$$

and

$$\begin{aligned} ({}_0\Delta_h^{-\alpha}\varphi_{k,s})(nh+a) &= \frac{1}{\Gamma(\mu+\alpha+1)}(nh+a+\mu h)_h^{(\mu+\alpha)} \\ &= \frac{\Gamma(\alpha+n-(k+1)+\mu+1)}{\Gamma(\mu+\alpha+1)\Gamma(n-(k+1)+1)}h^{\mu+\alpha} \\ &= \frac{\Gamma(\alpha+n-k+\mu)}{\Gamma(\mu+\alpha+1)\Gamma(n-k)}h^{\mu+\alpha} \\ &= \binom{n-k-1+\mu+\alpha}{n-k-1}h^{\mu+\alpha} \\ &= \binom{n-(k+1)+(k+1)\alpha+s\beta}{n-(k+1)}h^{(k+1)\alpha+s\beta} \\ &= \varphi_{k+1,s}(nh). \end{aligned}$$

□

From the application of the power rule follows the rule for composing two fractional h -sums. The proof for the case $h = 1$ one can find in [10]. For any positive $h > 0$ we presented the prove in [9].

Proposition 11. *Let x be a real valued function defined on $(h\mathbb{N})_a$, where $a, h \in \mathbb{R}, h > 0$. For $\alpha, \beta > 0$ the following equalities hold:*

$$\begin{aligned} \left({}_{a+\beta h}\Delta_h^{-\alpha}\left({}_a\Delta_h^{-\beta}x\right)\right)(t) &= \left({}_a\Delta_h^{-(\alpha+\beta)}x\right)(t) \\ &= \left({}_{a+\alpha h}\Delta_h^{-\beta}\left({}_a\Delta_h^{-\alpha}x\right)\right)(t), \end{aligned}$$

where $t \in (h\mathbb{N})_{a+(\alpha+\beta)h}$.

The next proposition gives a useful identity of transforming Caputo fractional difference equations into fractional summations for the case when an order is from the interval $(0, 1]$.

Proposition 12. [9] *Let $\alpha \in (0, 1]$, $h > 0$, $a = (\alpha - 1)h$ and x be a real valued function defined on $(h\mathbb{N})_a$. The following formula holds*

$$\left({}_0\Delta_h^{-\alpha}\left({}_a\Delta_{h,*}^{\alpha}x\right)\right)(nh+a) = x(nh+a) - x(a), \quad n \in \mathbb{N}_1.$$

3 Systems with sequential fractional differences

Let $\alpha, \beta \in (0, 1]$ and $x : (h\mathbb{N})_a \rightarrow \mathbb{R}^n$. Moreover, let us take $a = (\alpha - 1)h$ and $b = (\beta - 1)h$. Then we define

$$y(nh+b) := \left({}_a\Delta_{h,*}^{\alpha}x\right)(nh).$$

Note that $y : (h\mathbb{N})_b \rightarrow \mathbb{R}^n$. Then we apply the next difference operator of order β on the new function y and consider here an initial value problem stated by the system:

$$({}_a\Delta_{h,*}^\alpha x)(nh) = y(nh+b), \quad (7a)$$

$$({}_b\Delta_{h,*}^\beta y)(nh) = f(nh, x(nh+a)) \quad (7b)$$

with initial values:

$$({}_a\Delta_{h,*}^\alpha x)(0) = x_0, \quad (8a)$$

$$x(a) = x_a, \quad (8b)$$

where x_a, x_0 are constant vectors from \mathbb{R}^n . We use Proposition 12 twice and, for $n \geq 1$, we get:

$$\begin{aligned} ({}_0\Delta_h^{-\beta} ({}_b\Delta_{h,*}^\beta y))(nh+b) &= y(nh+b) - y(b) \\ &= ({}_a\Delta_{h,*}^\alpha x)(nh) - x_0 \end{aligned}$$

and

$$({}_0\Delta_h^{-\alpha} ({}_a\Delta_{h,*}^\alpha x))(nh+a) = x(nh+a) - x_a.$$

Hence

$$({}_a\Delta_{h,*}^\alpha x)(nh) = x_0 + ({}_0\Delta_h^{-\beta} \tilde{f})(nh+b), \quad (9)$$

where $\tilde{f}(nh) := f(nh, x(nh+a))$. Nextly

$$x(nh+a) = x_a + x_0 ({}_0\Delta_h^{-\alpha} 1)(nh+a) + ({}_0\Delta_h^{-\alpha} g)(nh+a), \quad (10)$$

where $g(nh) = ({}_0\Delta_h^{-\beta} \tilde{f})(nh+b)$.

Firstly we prove the formula for the unique solution in linear case, i.e. when in equation (7): $f(nh, x(nh+a)) = Ax(nh+a)$, where A is a constant square matrix of degree n .

Theorem 13. *The solution to the system*

$$({}_a\Delta_{h,*}^\alpha x)(nh) = y(nh+b), \quad (11a)$$

$$({}_b\Delta_{h,*}^\beta y)(nh) = Ax(nh+a) \quad (11b)$$

with initial conditions (8), i.e. $({}_a\Delta_{h,*}^\alpha x)(0) = x_0$ and $x(a) = x_a$, $x_0, x_a \in \mathbb{R}^n$, is given by the following:

$$x(nh+a) = \sum_{k=0}^{+\infty} A^k (\varphi_{k,k} x_a + \varphi_{k+1,k} x_0)(nh), \quad (12)$$

for $n \in \mathbb{N}_0$.

Proof. For $n = 0$ let us notice, that only $\varphi_{0,0}(0) = 1$, for any $k > 0$ the next terms are zero, so in fact we have: $x(0h + a) = x_a$.

For $n > 0$ let us define the following sequence

$$x_{m+1}(nh + a) = x_a \varphi_{0,0}(nh) + x_0 \varphi_{1,0}(nh) + ({}_0\Delta_h^{-\alpha} g_m)(nh + a), \quad m \in \mathbb{N}_0,$$

where $g_m(nh) = ({}_0\Delta_h^{-\beta} \tilde{f}_m)(nh + b)$ and $\tilde{f}_m(nh) = Ax_m(nh + a)$ with $x_0(nh + a) = x_a$.

We calculate the first step. As $\tilde{f}_0(nh) = Ax_0(nh + a) = Ax_a$, then $g_0(nh) = Ax_a ({}_0\Delta_h^{-\beta} 1)(nh + b) = Ax_a \varphi_{0,1}(nh)$. Going further,

$$x_1(nh + a) = x_a \varphi_{0,0}(nh) + x_0 \varphi_{1,0}(nh) + ({}_0\Delta_h^{-\alpha} g_0)(nh + a).$$

What could be written as

$$x_1(nh + a) = (x_a \varphi_{0,0} + x_0 \varphi_{1,0} + Ax_a \varphi_{1,1})(nh).$$

and, using Proposition 10, we get

$$x_2(nh + a) = (x_a \varphi_{0,0} + x_0 \varphi_{1,0} + Ax_a \varphi_{1,1} + Ax_0 \varphi_{2,1} + A^2 x_a \varphi_{2,2})(nh).$$

and the next element of the considered sequence has the following form:

$$x_3(nh + a) = (x_a \varphi_{0,0} + x_0 \varphi_{1,0} + Ax_a \varphi_{1,1} + Ax_0 \varphi_{2,1} + A^2 x_a \varphi_{2,2} + A^2 x_0 \varphi_{3,2} + A^3 x_a \varphi_{3,3})(nh).$$

Taking m tending to $+\infty$ we get formula (12) as the solution of (11) with initial conditions (8). \square

3.1 Semilinear sequential systems

Firstly we state some technical lemma and notations.

Lemma 14. *Let $u : (h\mathbb{N})_0 \rightarrow \mathbb{R}$ and $\alpha > 0$. Let $({}_0\Delta_h^{-k\alpha} \gamma)(nh + k\alpha h) = \gamma_1(nh + k\alpha h)$ and $\tilde{\gamma}_1(nh) := \gamma_1(nh + k\alpha h)$ for $k \in \mathbb{N}$. Then for $k \in \mathbb{N}$ we get*

$$({}_0\Delta_h^{-\alpha} \tilde{\gamma}_1)(t) = ({}_0\Delta_h^{-(k+1)\alpha} \gamma)(t + k\alpha h), \quad (13)$$

where $t = nh + \alpha h$.

Proof. First let us consider the case $k = 1$. Then from Proposition 11 we can write

$$({}_\alpha h \Delta_h^{-\alpha} ({}_0\Delta_h^{-\alpha} \gamma))(t) = ({}_0\Delta_h^{-2\alpha} \gamma)(t),$$

where $t = nh + 2\alpha h$, $n \in \mathbb{N}_0$.

Let $\gamma_1(nh + \alpha h) = ({}_0\Delta_h^{-\alpha}\gamma)(nh + \alpha h)$ and $\tilde{\gamma}_1(nh) := \gamma_1(nh + \alpha h)$. Then

$$\begin{aligned}
({}_0\Delta_h^{-\alpha}\tilde{\gamma}_1)(nh + \alpha h) &= \\
&= \frac{h}{\Gamma(\alpha)} \sum_{r=0}^n (nh + \alpha h - \sigma(rh))_h^{(\alpha-1)} \tilde{\gamma}_1(rh) \\
&= \frac{h}{\Gamma(\alpha)} \sum_{s=\alpha}^{n+\alpha} (nh + 2\alpha h - \sigma(sh))_h^{(\alpha-1)} \gamma_1(sh) \\
&= ({}_{\alpha h}\Delta_h^{-\alpha}\gamma_1)(nh + 2\alpha h) \\
&= ({}_0\Delta_h^{-2\alpha}\gamma)(nh + 2\alpha h).
\end{aligned}$$

The equation (13) for $k > 1$ follows inductively. \square

Note that

$$\begin{aligned}
({}_0\Delta_h^{-k\alpha}\gamma)(nh + k\alpha h) &= \\
&= \frac{h}{\Gamma(k\alpha)} \sum_{r=0}^n (nh + k\alpha h - \sigma(rh))_h^{(k\alpha-1)} \gamma(rh).
\end{aligned}$$

Similarly to the procedure presented in the proof of Lemma 14 we can prove that for $k, s \in \mathbb{N}_0$ and $\alpha > 0, \beta > 0$:

$$\begin{aligned}
({}_0\Delta_h^{-k\alpha-s\beta}\gamma)(nh + k\alpha h + s\beta h) &= \\
&= \frac{h}{\Gamma(k\alpha + s\beta)} \sum_{r=0}^n ((n + k\alpha + s\beta - r - 1)h)_h^{(k\alpha+s\beta-1)} \gamma(rh) \\
&= h^{k\alpha+s\beta} \sum_{r=0}^n \frac{\Gamma(n - r + k\alpha + s\beta)}{\Gamma(k\alpha + s\beta)\Gamma(n - r + 1)} \gamma(rh) \\
&= \sum_{r=0}^n \binom{n - r + k\alpha + s\beta - 1}{n - r} h^{k\alpha+s\beta} \gamma(rh). \quad (14)
\end{aligned}$$

Taking $\mu = k\alpha + s\beta$ and using formula (4) we can write (14) shortly in the following way

$$({}_0\Delta_h^{-\mu}\gamma)(nh + \mu h) = \sum_{r=0}^n \tilde{\varphi}_{k,s}(nh - rh) \gamma(rh). \quad (15)$$

Moreover, we can also write direct formula for values $({}_0\Delta_h^{-\alpha}g)(nh + a)$ given in (10) for nonlinear problem. In fact using Definition 3 of fractional summation, formula (4) of functions $\tilde{\varphi}_{k,s}$ and Proposition 11 we write (15) as follows:

$$x(nh + a) = x_a + x_0 ({}_0\Delta_h^{-\alpha}1)(nh + a) + \sum_{r=0}^n \tilde{\varphi}_{1,1}(nh - \sigma(rh)) f(rh, x(rh + a)).$$

Using the power rule formula for $\mu = 0$ and by Remark 9 we can write the recursive formula for the solution to nonlinear problem given by equations (7) and conditions (8):

$$x(nh + a) = x_a + x_0\varphi_{1,0}(nh) + \sum_{r=0}^n \tilde{\varphi}_{1,1}(nh - \sigma(rh))f(rh, x(rh + a)). \quad (16)$$

The given formula (16) also works for $n = 0$ as $\tilde{\varphi}_{1,1}(-h) = 0$. Then $x(0h + a) = x_a$. We can also check for example the next step:

$$x(h + a) = x_a + x_0\varphi_{1,0}(h) + \tilde{\varphi}_{1,1}(0h)f(0, x(a)) = x_a + x_0h^\alpha + h^{\alpha+\beta}f(0, x(a)).$$

For special semilinear case when $f(nh, x(nh + a)) = Ax(nh + a) + \gamma(nh)$ we have $f(0, x(a)) = Ax(a) + \gamma(0)$. Then

$$x(h + a) = (I + h^{\alpha+\beta}A)x_a + h^\alpha x_0 + h^{\alpha+\beta}\gamma(0).$$

Theorem 15. *The solution to the system*

$$({}_a\Delta_{h,*}^\alpha x)(nh) = y(nh + b), \quad (17a)$$

$$({}_b\Delta_{h,*}^\beta y)(nh) = Ax(nh + a) + \gamma(nh) \quad (17b)$$

with initial conditions (8), i.e. $({}_a\Delta_{h,*}^\alpha x)(0) = x_0$ and $x(a) = x_a$, $x_0, x_a \in \mathbb{R}^n$ is given by

$$x(nh + a) = \sum_{k=0}^{+\infty} A^k (\varphi_{k,k}x_a + \varphi_{k+1,k}x_0)(nh) + \sum_{r=0}^n \left(\sum_{k=0}^{\infty} A^k \tilde{\varphi}_{k+1,k+1}(nh - \sigma(rh)) \right) \gamma(rh), \quad (18)$$

for $n \in \mathbb{N}_0$.

Proof. For $n = 0$ let us notice, that only $\varphi_{0,0}(0) = 1$, for any $k > 0$ the next terms are zero, so in fact we have: $x(0h + a) = x_a$.

For $n > 0$ based on the proof for linear case we can write the solution formula as follows:

$$x(nh + a) = \sum_{k=0}^{+\infty} A^k (\varphi_{k,k}x_a + \varphi_{k+1,k}x_0)(nh) + \sum_{k=0}^{+\infty} A^k \left({}_0\Delta_h^{(k+1)(\alpha+\beta)} \gamma \right) (nh - h + (k+1)(\alpha + \beta)h).$$

Then taking into account the formulas (14) and (15) we get the form (18) as the solution of (17) with initial conditions (8). \square

4 Positivity

Based on [12, 13] we consider the following definitions.

Definition 16. The fractional system (7) is called *positive* fractional system if and only if $x(nh + a) \in \mathbb{R}_{\geq}^n$ for any initial conditions $x_a, x_0 \in \mathbb{R}_{\geq}^n$.

Let $x_0, x_a \in \mathbb{R}_{\geq}^n$ and $\mathcal{X}_n := \{X : \mathbb{N} \rightarrow \mathbb{R}^n\}$. The operator $T_{x_0, x_a} : \mathcal{X}_n \rightarrow \mathcal{X}_n$ is defined as

$$(T_{x_0, x_a} X)(n) := \sum_{k=0}^{+\infty} A^k (\varphi_{k,k} x_a + \varphi_{k+1,k} x_0)(nh) \\ + \sum_{r=0}^n \left(\sum_{k=0}^{\infty} A^k \tilde{\varphi}_{k+1,k+1}(nh - \sigma(rh)) \right) \gamma(rh),$$

where $X \in \mathcal{X}_n$, $X(n) = x(nh + a)$ and functions $\varphi_{k,s}, \tilde{\varphi}_{k,s}$ are given by (3) and (4), respectively.

The proof of the following proposition is analogous to similar fact in [12].

Proposition 17. Let $x_0, x_a \in \mathbb{R}_{\geq}^n$ and the right hand side of system (7), f fulfils $f(nh, x(nh + a)) \geq 0$. Then $T_{x_0, x_a}(\mathbb{R}_{\geq}^n) \subset \mathbb{R}_{\geq}^n$.

Definition 18. The fractional system (7) is called *locally positive fractional system* if and only if for any initial conditions $x_a, x_0 \in \mathbb{R}_{\geq}^n = \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n\}$ there is $\tau \geq 1$ such that $x(nh + a) \in \mathbb{R}_{\geq}^n$ for $n \in [0, \tau]$.

Proposition 19. If the matrix $I + Ah^{\alpha+\beta}$ is positive and $\gamma(nh) \geq 0$ for $n \in \mathbb{N}_0$, then for any $x_0, x_a \in \mathbb{R}_{\geq}^n$ the fractional system (17) is locally positive.

Proof. Since $x(h + a) = \sum_{k=0}^1 A^k (\varphi_{k,k} x_a + \varphi_{k+1,k} x_0)(h) = x_a + \varphi_{1,0}(h)x_0 + A\varphi_{1,1}(h)x_a = x_a + h^\alpha x_0 + Ah^{\alpha+\beta} x_a$, then with our assumptions, we get local positivity. □

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